Pairwise Influences and Bargaining Among the Many*

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Abstract

In this paper we build a model to analyze distributional conflict when the utility of each agent is heterogeneously influenced by other agents' utility. There are two dimensions of heterogeneity: who influences whom, and the strength of any such pairwise dependent influence. The pattern of bilateral influences takes the form of a weighted and directed network. Direct influences spread their effects through chains of connected agents in this network. We characterize the Nash bargaining solution and analyze how pairwise influences, and the indirect effects they generate, are internalized in shares and utilities obtained. The analysis relies on network centrality indexes that measure each agent's prominence due to his position in the influences structure. Our results have implications for the study of urban crime, government spending, and bargaining with social preferences.

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1 Introduction

Influence in outcomes and behavior across individuals is pervasive in many social and economic settings. This influence can take many forms. It can be centralized, and result from a common influence of the environment. Else, it can be decentralized and supported by more local interactions. At its most disagreggated level, influence is mediated by pairwise interactions. The collection of such pairwise interactions can display a rich pattern that might overall influence the resulting individual outcomes.

Cross influences can be an important issue if we deal with distributional conflict. Consider a fixed resource to be divided among a set of individuals. Any possible division of this resource has a direct effect on individual outcomes. In addition to this direct effect, there can also be an indirect effect mediated by the possible interactions that might arise across some pairs of individuals. Through the chains of interactions present inside the group, these indirect effects can spread all over the population. When individuals agree collectively on the division of this common resource, all these influence effects have to be internalized adequately. The pattern of direct pairwise influences can thus play a crucial role in the solution of the distributional conflict.

When considering disaggregated pairwise influences, there are two different possible sources of heterogeneity in its underlying structure: the geometry of interactions, that expresses who influences whom, and the magnitude of each one of these influences. The aim of this paper is to analyze how the whole structure of such pairwise influences affects the bargaining outcomes.

Consider for example the case of urban crime. Crime patterns on city's neighbourhoods are interlaced and there are several reasons why this is so.

First, the division of cities into different neighbourhoods is merely an administrative issue. Criminals do not doubt in crossing the street and move from a neighbourhood to another to commit their activities. Hence, if no natural barrier imposes difficulties to this movement, this is a natural source of interdependence in crime rates across neighbourhoods. This is very much related with the so called Modifiable Area Unit Problem (M.A.U.P.) highlighted in the literature of spatial statistics in geographic information science. Different groupings into neighbourhoods lead to variability in statistical results. When aggregating for crime rates it is unlikely that there exists a univocal relation between neighbourhood characteristics and crime outcomes.

Furthermore, the embeddedness of individuals into social networks that might spread over the city, impacts also criminal behavior. Some individuals might imitate the behavior of their contacts in their network of acquaintances (see Glaeser et al., 1996). Some others might infer (maybe erroneously) information on the benefit on commiting crime (see Sah, 1993). Hence, criminal behavior suffers a contagion effect inlaid to the social environment. This social osmosis process also induces spatial influence in crime rates.

When the major has to decide on how to distribute resources to fight urban crime, and when a representative of each neighborhood asks for part of these resources, we come up with a distributional conflict with influences on crime rates across pairs of neighborhoods. Our analysis sheds light on how the particular neighborhood geography and the pattern of influences across neighbourhoods are internalized in the final resource allocation.

The environment. There are two dimensions of heterogeneity in the model. On the one hand, the particular geometry of bilateral influences. Not everybody necessarily exerts an influence on every other agent. The pattern of direct bilateral influences determines who exerts an externality on whom and how influences spread indirectly through the economy. On the other hand, the magnitude of each bilateral influence is pairwise dependent. The magnitude of an influence relation depends on exactly which agent exerts this influence and which agent receives it.

Direct influences generate indirect network effects. We call network externalities the sum of all these indirect effects. These are direct externalities derived from the allocation of resources, and measure how the level of consumption, not the utility, of an agent affects another agent's utility. Each pattern of pairwise influences determines a unique pattern of network externalities.

To better understand the spread of influences through direct bilateral influences we reinterpret at some points in the paper the model in terms of networks. The network links an agent to another whenever the second agent exerts a direct externality on the first one. Externalities spread then through the links of the network. We can keep track of all indirect influences generated from bilateral influences through paths and cycles in the associated network. The network metaphor is adequate in this setting due to asymmetric pairwise bilateral influences. Connections with well-known notions from social network analysis, such as centrality measures, arise as natural tools for the network reinterpretation of the model.

Direct bilateral influences within a finite set of agents are modeled with the use of a linear model that encompasses the possibility of positive and negative influences as well as asymmetric influences within pairs of agents. The model is characterized by a matrix that collects the possibly different levels of bilateral influences for each possible pair of agents, the primitives of the model.

A unit of a divisible resource has to be distributed among the agents of the economy. The utility that an agent obtains comes from the share of the resource that receives and also from the utility of agents that exert an influence on him, which enters his utility function proportionally to the intensity of the direct influence each of these other agents exert on him.

In the example on crime, whenever direct influences exist among crime rates in different neighbourhooods, the allocation of a part of the resources against urban crime does not only have a direct effect on this neighbourhood. Its effects also spill over the rest of neighbourhoods in an heterogeneous manner. Hence, when the different neighbourhoods are engaged in dispute for these resources, and they understand that these spillover effects exist, they show heterogeneous preferences on possible assignments. Even if probably each neighbourhood would prefer to receive all the resource, if they realize they have to achieve an agreement with the rest of neighbourhoods, they would prefer that those neighbourhoods that overall exert a larger spillover effect on it receive more resources than those that exert less.

A linear structure of influence is assumed for tractability. It ensures for, almost, every economy with influences the existence of a unique solution to the system relating utilities. This solution provides the utility in terms of the allocation, instead of in terms of others' utilities. Hence, solving the model means characterizing the effect the share of the resource an agent receives changes others' welfare, and it internalizes the indirect effects bilateral influences generate.

Still another assumption has to be argued. We suppose constant returns to the share of the resource received. This is an strong assumption. Probably the returns to direct investment on fighting against crime in a neighborhood are not constant. However, assuming constant returns, influences become the unique underlying force that yield to differences in the results. In general, the introduction of concavities due to other reasons not related to influences would distort the analysis of the effect of influences on the bargaining outcome.

When distributional conflict exists, the bargaining outcome is given by the Nash bargaining solution. With its use we ensure a unique well-defined outcome after the resolution of the conflict. Of course, this choice comes at a cost. By using a cooperative solution we abstract from institutional or environmental restrictions that can play a role during the bargaining process. These institutional restrictions are generally introduced defining a particular non-cooperative bargaining game that takes them into account. Then each particular application should be followed by a different non-cooperative game that specifies its particularities. This way we would lose some of the general conclusions that a more stylized model can give as general features of somewhat different situations.

Results. We restrict our analysis to the more interesting case of regular economies. Regular economies are such that any allocation that exhausts resources is Pareto efficient. In regular economies, the Pareto frontier is non-degenerate.

For such economies, we characterize completely the Nash bargaining solution, providing closed-form expressions for the utilities and shares obtained.

The reason for which we restrict to regular economies is the following. In non regular economies the Pareto frontier is degenerate. This is so because the influence exerted by some given agent on others is much bigger than the influence he receives in return. Efficiency might require then that this agent receives all the resources available. If an individual in a group is so much loved by anybody else compared with other possible affective relations, which means that the influence this agent has on others' utility is very large, it could be efficient to give him all the resource.

Similarly, there might exists intermediate situations in which only some agents are allowed to receive a share of the resource for efficiency reasons. While our methods and analysis could be extended to such nonregular situations, we focus on the analysis of regular economies in which the pattern of influences excludes *de facto* some agents from the course for some part of the resources.

We first characterize regular economies. This characterization is twofold. First, regular economies are characterized by an upper bound on the aggregate level of bilateral influences every agent exerts on others. In terms of direct bilateral influences, we obtain a bound on the maximal level of aggregate direct influences an agent can exert. The economy is regular if and only if no agent exceeds this bound. Second, regular economies are characterized by conditions on the pattern of network externalities. More precisely, an economy is regular if and only if all agents are equally central in the network structure of influences. The relevant measure of network centrality is the Katz-Bonacich centrality index, pervasively used in the sociology literature, and that also arises naturally in other economic settings.

We next provide a constructive procedure to characterize the Nash bargaining solution. It is important to note that even if the economy is regular and, hence, the distributional conflict involves all agents in the economy, this does not exclude the possibility that some agents obtain finally no share of the resource. Externalities do not directly solve the bargaining problem but this does not mean that they can not be sufficiently asymmetric such that, after internalizing all influences, the Nash bargaining solution assigns nothing to some of the agents.

A geometric procedure is presented to check if the solution for a particular economy is interior, meaning all agents obtain a positive fraction of the resource, and to characterize the solution in this case.

We also devote part of our work to the analysis of α-economies. In these economies all existent pairwise influences have intensity equal to α, and whenever an agent exerts an influence on another, this other agent also exerts an influence on the first one. One of the two dimensions of heterogeneity in the model, the possibly different levels of externality intensities across individuals, is kept to a minimum. The main source of heterogeneity is the geometry of the pattern of pairwise influences.

An analysis in depth of this family of economies gives a better picture of how the particular arrangement of pairwise relations, irrespective of the intensities of these, impacts on bargaining outcomes. In particular, utility is directly related with the number of connections an agent has. Those agents that receive and exert more influences are also rewarded with larger utility levels. However, this monotonicity does not necessarily translates into receiving larger fractions of the resource.

Finally, we also study how changes on the pattern of influences distort the Nash bargaining outcome. We analyze how changes in the levels of bilateral influences change the bargaining result. Furthermore, we also discuss how our framework can be used to describe and analyze situations in which some agents that are not involved in the bargaining game can affect the bargaining outcome.

Applications. Besides the example on urban crime, our model is flexible enough to encompass other possible applications.

Consider for example government spending. When deciding how to divide the public budget within government departments a possible concern for the ministers is to take into account that outcomes related to responsibilities of one department can affect outcomes related to other departments. The result of the bargaining process should then depend on these influences generated across departments.

The department of social affairs might exert a positive externality on the department of education. If more resources are spent on social affairs, such as ameliorating life conditions in specially poor regions, this can translate into larger school attendance rates in this regions and hence an improvement in aggregate level of education of the country, a subject under the domain of the education department. Moreover, this effect on citizens' education can translate into an increase on the understanding of good and healthy habits that might imply an increase on life expectancy in the country, a fundamental issue for the health department.

Observe that the outcome improvement on health comes indirectly from an increase of resources for social affairs. This increase on resources improves the outcome for social affairs which through a direct influence improves the outcome on education, which improves the outcome on health. This is another example of indirect influences that spread due to network effects. In this setup our analysis sheds some light on how the pattern of influences across different departments maps into government bargaining agreements.

Another possible application is in the field of social preferences.[1] Following the line of seminal work from Gary Becker (1974), altruism and envy can be interpreted as influences of a very particular kind, with its source coming from psychological reasons. An agent is altruist for another if he is better off when the other is better off. Hence, this second agent is exerting a positive influence on the first one. Inversely, an agent is envious for another agent if he is worst off when the other is better off, and in this case the influence the second agent exerts on the first is negative. Our work applies then to the analysis of a bargaining game in the presence of pure altruism and envy effects, where the pattern of altruism and envy is variable in intensity across pairs of agents.

Related Literature. Our model bears a formal resemblance with previous work on interdependent utilities by Bergstrom (1999) and Bramoullé (2001). Bramoullé also interprets this type of systems in terms of weighted and directed networks, but focuses on some qualitative features of the mapping from bilateral influences to network externalities.[2] Here instead, we analyze the mapping from bilateral influences to Nash bargaining utilities and agreed shares, providing closed-form expressions for both.

[1]See Fehr and Schmidt (2002) and Sobel (2005) for very comprehensive surveys of the theoretical literature and the empirical evidence on social preferences. Levine uses a linear model similar to the one we develop in our work to analyze experimental results for some classes of games. Roth (1995) is a survey of experimental evidence of social preferences in bargaining games.

[2]For a complementary approach to this mapping and a general characterization of Pareto efficiency in such a setup, see de Martí (2006).

A strategic model on status in networks that generates similar interdependency systems is provided in Rogers (2005). Rogers analyzes a network formation game in which agents with heterogeneous skills can choose with whom they want to contact and with which intensities they want that this contact is made. Hence, the pattern of influences is endogenously chosen. We analyze instead situations in which the structure of pairwise influences can not easily be affected by individual strategic decisions. For example, in the urban crime example no neighbourhood can do much to delimit influences among them, since these are largely determined by private decisions and actions of the population, which is an issue out of their control. A similar comment applies on the example on interdepartmental influences in a government. Also, altruism and envy are not only the result of strategic decisions but the effect of the embeddedness of individuals on a social environment they can not necessarily determine and control.

The model also resembles input-output models of linear economies (Leontief, 1951, Gale, 1960). However, input-output models only allow for positive bilateral influences, while here we do not impose sign restrictions of any sort. Of course, we also deal with very different issues.

Some papers have analyzed multilateral bargaining with externalities from a non-cooperative viewpoint. Jehiel and Moldovanu (1995a, 1995b) consider a setup where one seller bargains with n potential buyers to decide which of them obtains the unit of an indivisible good. The acquisition of the good by one of the agents can exert a positive or negative externality on others. They analyze how the bargaining outcome is affected by this allocative externality.

In a political economy context, Calvert and Dietz (2004) explore how the introduction of externalities in a 3-agent economy alters the conclusions of the Baron and Ferejohn (1989) non-cooperative game of legislative bargaining. See Duggan (2004) for conditions about existence of equilibria in the n agents version of the Baron and Ferejohn game with externalities.

While our cooperative approach is less sensitive to possible particularities in the bargaining process, such as the particular mechanisms by which buyers and sellers bargain or the existence of a voting rule (the majority rule in Baron-Ferejohn models) in legislative bargaining, it allows for a general and tractable analysis of multilateral bargaining with one unique outcome prediction and an heterogeneous pattern of externalities.

Our work also borrows from the very active literature on networks in economics. However, we do not deal with the formation of social and economic networks, maybe the more extensively studied issue in the field, but on games played in a fixed network.[3] Other authors have also explored the interrelation of network structure and bargaining outcomes (see Calvó-Armengol, 2001, and Corominas-Bosch, 2004). The approach in these papers is different in many respects. Just to mention a few, bargaining is not among the many and the network represents communication restrictions and delimits the possible pairs of agents that can trade.

[3]See Jackson (2005) for a very extensive survey of the field of networks in economics, and for an exhaustive list of references about games played in networks, including bargaining games.

The Katz-Bonacich centrality measure was first defined by Katz (1953) and later on developed by Bonacich (1987). It is one of the more relevant centrality measures studied in the active field of social network analysis.[4] Another game played in a network, in this case not a bargaining game, where this centrality measure naturally arises is Ballester et al. (2006). Agents play a game with pairwise dependent strategic complementarities. In the unique equilibrium of the game each agent action is proportional to his Katz-Bonacich centrality index measured on this network of complementarities.

2 Bilateral Influences and Network Externalities

2.1 Modelling Bilateral Influences

In this section we propose a simple framework that allows to model positive and negative allocative influences across individuals.

Suppose that there is an amount of a certain resource to be distributed within a group of n individuals, $\mathcal{N} = \{1, \ldots, n\}$. Let c_i be the consumption of agent $i \in \mathcal{N}$. Let $b_{ij} \in \mathbb{R}$ be the magnitude of the influence agent j exerts on agent i. Then, an increase of one unit of welfare for agent j induces an increase of b_{ij} units of welfare for agent i. Given a profile $\mathbf{c} = (c_1, \ldots, c_n)$, the utility an agent obtains , $u_i(\mathbf{c})$, is equal to

$$u_i(\mathbf{c}) = c_i + \sum_{j \neq i} b_{ij} u_j(\mathbf{c}) \qquad i = 1, \ldots, n \tag{1}$$

This set of equations forms what we call the bilateral influences system. Note that the relation in this system is from outcomes to outcome.

In terms of the urban crime example, b_{ij} represents how the crime rate , not the share of public budget received, in neighborhood j affects the crime rate in neighborhood i.

Due to linearity, we can fix the sum of consumption levels to $\sum_{i=1}^{n} c_i$ to be equal to 1. Hence, we can interpret c_i as the share of the resource received by agent i.

Defining $b_{ii} = 0$ for all $i \in \mathcal{N}$, we gather all the b_{ij} in a matrix $\mathbf{B}$ of bilateral influences. An *economy* is completely characterized by its matrix of bilateral influences.

For a given economy $\mathbf{B}$, we can obtain from the structural system of bilateral influences to a reduced-form system where the utility of each agent can be directly expressed in terms of the shares profile, eliminating the dependency on other's utility.

The bilateral influence system in matrix form is equal to

$$\mathbf{u}(\mathbf{c}) = \mathbf{c} + \mathbf{B} \cdot \mathbf{u}(\mathbf{c}) \tag{2}$$

Hence, if $\mathbf{I}$ is the identity matrix, whenever $(\mathbf{I} - \mathbf{B})^{-1}$ exists we obtain the reduced-form system

$$\mathbf{u}(\mathbf{c}) = (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{c} \tag{3}$$

[4]For an exhaustive survey of this literature see Wasserman and Faust(1994).

The first result we provide is a genericity result. An economy is characterized by $n(n-1)$ real values. Therefore, there is a one-to-one mapping from economies to elements of $\mathbb{R}^{n(n-1)}$. From this point of view, the set of economies in $\mathbb{R}^{n(n-1)}$ for which $(\mathbf{I}-\mathbf{B})^{-1}$ does not exist has (Lebesgue) measure zero. This implies that for almost every economy $\mathbf{B}$, the associated matrix $(\mathbf{I}-\mathbf{B})^{-1}$ exists (and, of course, is unique) and the next result then follows.

Proposition 1 *For almost every economy* $\mathbf{B}$ *the associated reduced-form system is uniquely characterized.*

In words, given a structural system of bilateral influences there is no indeterminacy in the obtention of the associated reduced-form expression, except for a negligible set of economies.[5]

Let $\mathbf{E}(\mathbf{B}) = (\mathbf{I}-\mathbf{B})^{-1}$. Each entry $e_{ij}(\mathbf{B})$ expresses the magnitude of how the utility increases, if the entry is positive, or decrease, if the entry is negative, when the level of consumption of agent j increases. We call $\mathbf{E}$ the matrix of *network externalities*. An explanation for the choice of this name follows.

2.2 From Bilateral Influences to Network Externalities

Any economy $\mathbf{B}$ can be naturally represented by a network.

A network is formed by a set of nodes and a set of links that express a relation between the pair of nodes linked. While this is an abstract object, it is a useful metaphor to represent many varied situations in applied settings. In particular, in our case this metaphor can be applied to make nodes represent the agents involved in the structural system of bilateral influences, and make links represent the pattern of bilateral influences exerted across pairs of agents. A link in such an influence network is weighted, each link has an associated value that represents the strength of the influence this link represents, as well as a particular direction, since the influence agent i exerts on j does not necessarily coincides in strength with the influence agent j exerts on agent i, and hence we have to distinguish the link from i to j and the link from j to i.

Different conventions could be adopted to express the mapping from economies to networks. We adopt the following one. We say that there is a link from agent i to agent j whenever j exerts a, positive or negative, influence on i, and the weight for this link is then equal to the coefficient $b_{ij} \in \mathbb{R}$ of the structural system of bilateral influences. Since in our model there is no self-influence we do not allow for self-loops, links from an agent to itself. The set links that begin in i point to the agents that influence agent i.

Observe the weighted and directed nature of the network defined in this way: since we have not imposed any restriction on the possible values of the coefficients in the structural influence system, the weight of a link can take any real value; also, since we have not imposed symmetry on the levels

[5]See Bramoullé(2001) for structural models of a similar nature for which indeterminacy in the determination of the associated reduced-form system arises.

of bilateral influence, it is possible that there exist both a link from i to j and another one from j to i and that their respective weights differ. Even more, it is possible that there exist a link from i to j while there is no link from j to i.

A weighted and directed network is defined by an adjacency matrix, where the entry (i, j) in this matrix is equal to the weight of the link from i to j. This weight equals the level of bilateral influence j exerts on i. Hence, given an economy $\mathbf{B}$ the adjacency matrix of its associated network, in the way we have defined this network, is also $\mathbf{B}$.

The following equality applies

$$\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1} \tag{4}$$

Whenever $\mathbf{B}$ is a contraction[6] we have that

$$(\mathbf{I} - \mathbf{B})^{-1} = \sum_{k=0}^{+\infty} \mathbf{B}^k \tag{5}$$

If j exerts an influence on i with weight b_{ij} and k exerts an influence on j with weight b_{jk}, k exerts an indirect influence on i with weight equal to $b_{ij}b_{jk}$. The matrix $\mathbf{B}^2$ keeps track of these second order network influences. The entry $b_{ik}^{[2]}$ of $\mathbf{B}^2$ computes the sum of weights of all paths of length two from i to k.[7]

More generally, for any $l \geq 1$ the matrix $\mathbf{B}^l$ keeps track of the l-order network influences: each entry $b_{ik}^{[l]}$ equals the sum of weights of all paths of length l from i to k.

Therefore, whenever the expression in equation (1) is valid, the entry $e_{ij}(\mathbf{B})$ of $\mathbf{E}(\mathbf{B})$ is the sum of weights of *all* paths from j to i in the network represented by the economy/adjacency matrix $\mathbf{B}$. The matrix $\mathbf{E}(\mathbf{B})$ computes the sum of indirect (network) effects that the pattern of bilateral influences generates. This sum of indirect effects of any order is what we denote *network externalities*, and this is why we call matrix $\mathbf{E}(\mathbf{B})$ the matrix of *network externalities*.

Each entry $\mathbf{e}_{ij}(\mathbf{B})$ represents by how much the consumption of agent j affects the utility of agent i not only through the direct bilateral influence agent j exerts on i, represented by b_{ij}, but also through the indirect influences resulting of all possible indirect network connections from j to i.

The following example, borrowed from Bramoullé (2001), is useful to understand how important are indirect network effects for the analysis of the mapping from allocations to utilities defined in the reduced-form system.

[6]The matrix $\mathbf{B}$ is a contraction if and only if all its eigenvalues have norm smaller than 1. This will be the case for example for the set of *regular* economies, that we define later, if bilateral influences are positive.

[7]A *path* between i and j in network $\mathbf{G}$ is a sequence of agents $i_1, \ldots, i_K$ of N, where an agent can appear several times in this sequence, such that $i_k i_{k+1}$ is a link of $\mathbf{G}$ for every $k \in 1, \ldots, K-1$, with $i_1 = i$ and $i_K = j$. The length of such a path is equal to $K-1$, the number of links that form the path. In words, a path in g is an indirect connection from agent i to agent j through linked agents in $\mathbf{B}$. We define the *weight* of a path $i_1, \ldots, i_K$ of $\mathbf{G}$ as the product $g_{i_1 i_2} \cdots g_{i_{K-1} i_K}$. This weight is different than zero because of the definition of path. A path such that $i = j$ is called a *cycle*.

Example 1. There are three agents, $\mathcal{N} = \{1, 2, 3\}$, and the structural system of bilateral influences relating them is

$$\begin{aligned} u_1(\mathbf{c}) &= c_1 + b_{12}u_2(\mathbf{c}) + b_{13}u_3(\mathbf{c}) \\ u_2(\mathbf{c}) &= c_2 + b_{23}u_3(\mathbf{c}) \\ u_3(\mathbf{c}) &= c_3 \end{aligned}$$

where b_{12} and b_{13} are positive but b_{23} is negative. This means that both agent 2 and 3 exert a positive direct influence on agent 1, probably with different intensities, while agent 3 exerts a negative influence on agent 2. Besides, the values for these direct bilateral influences are b_{21}, b_{13} and b_{23}.

The network that represents this situation is

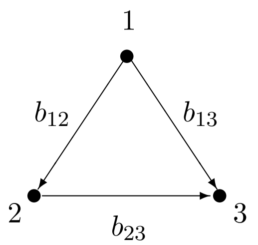


Figure 1

and the 3×3 bilateral influences matrix is

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the 3×3 matrix of network externalities, $\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1}$, is equal to

$$\mathbf{E}(\mathbf{B}) = \begin{pmatrix} 1 & b_{12} & b_{13} + b_{12}b_{23} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that in this case we can easily compute the matrices of indirect network effects, $\mathbf{G}^k$ for $k \geq 2$. The matrices of higher order network effects are

$$\begin{aligned} \mathbf{B}^2 &= \begin{pmatrix} 0 & 0 & b_{12}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \mathbf{B}^k &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \textit{for every } k \geq 3 \end{aligned}$$

Therefore, $\mathbf{E}(\mathbf{B}) = \mathbf{I}+\mathbf{B}+\mathbf{B}^2$. The expressions of utilities in terms of consumption are therefore

$$\begin{aligned} U_1(\mathbf{c}) &= c_1 + b_{12}c_2 + (b_{13} + b_{12}b_{23})\, c_3 \\ U_2(\mathbf{c}) &= c_2 + b_{23}c_3 \\ U_3(\mathbf{c}) &= c_3 \end{aligned}$$

The weight of the network externality agent 3 exerts on agent 1, $e_{13} = b_{13} + b_{12}b_{23}$, depends on the intensities of bilateral influences. In particular, if $-b_{12}b_{23} > b_{13}$ the externality is negative, even if the direct bilateral influence b_{13} is positive. A tension arises because even if agent 3 exerts a direct positive influence on agent 1, the negative influence agent 3 exerts on agent 2 also has an indirect (network) effect on agent 1 due to the indirect path from 3 to 1 through agent 2. This negative influence is internalized in the reduced-form system related to network externalities and makes it possible that e_{13} is negative if the bilateral negative influence 3 exerts on 2 is large enough.

We omit the dependence of $\mathbf{E}$ on $\mathbf{B}$ when no confusion is possible.

3 The Set of Pareto Allocations

3.1 Characterization

From now on we will consider that there is a certain amount of a resource that, without loss of generality, we normalize to one. Before turning to the study of distributional conflict and how agents in an economy agree to divide this unit of resource among them, we have to make a clarification about the set of Pareto efficient allocations in an economy with influences. Externalities can have severe consequences on which allocations can be Pareto efficient. Our aim in this section is to characterize the set of economies for which distributional conflict is particularly strong.

Before providing an example of the peculiar situations that can arise in economies with influences we describe the utility possibility set for any economy $\mathbf{B}$, that we denote $\mathrm{UPS}(\mathbf{B})$. Given an economy $\mathbf{B}$, and for any feasible allocation, we have that $\mathbf{u}(\mathbf{c}) = \mathbf{E}\cdot\mathbf{c} = \sum_{i=1}^n c_i \mathbf{e}^{(i)}$, where $\mathbf{e}^{(i)}$ is the i-th column vector of the matrix of network externalities. Since an allocation $\mathbf{c}$ is feasible if and only if $c_i \geq 0$ for every $i \in \mathcal{N}$ and $\sum_{i=1}^n c_i \leq 1$, we can conclude that the utility possibility set for the economy defined by $\mathbf{B}$ is the convex hull of the columns of the matrix of network externalities $\mathbf{E}$ plus the zero vector, that is

$$\mathrm{UPS}(\mathbf{B}) = co\left\{\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(n)}, \mathbf{0}\right\}$$

This implies that the utility possibility set for any economy $\mathbf{B}$ is a simplex, and therefore it is a convex and compact set.

Example 2. Consider the economy represented by the following network

$$\begin{tikzcd} 1 \arrow[r, shift left, "3/2"] & 2 \arrow[l, shift left, "1/2"] \end{tikzcd}$$

Figure 2

The 2×2 matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} 0 & 3/2 \\ 1/2 & 0 \end{pmatrix}$$

It follows that the matrix of network externalities for this economy is equal to

$$\mathbf{E}(\mathbf{B}) = \begin{pmatrix} 4 & 6 \\ 2 & 4 \end{pmatrix}$$

Here $\mathbf{e}^{(1)} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{e}^{(2)} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and the utility possibility set for this economy is the convex hull of these two vectors and the zero vector. We can depict $\mathrm{UPS}(\mathbf{B})$

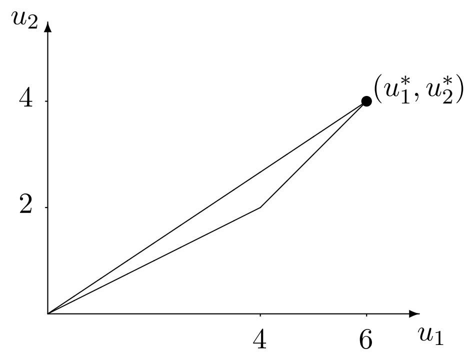


Figure 3

It turns out that the unique efficient allocation in this economy is (u_1^*, u_2^*), that corresponds to $(c_1^*, c_2^*) = (0, 1)$.[8] At the unique Pareto efficient allocation agent 2 receives all the resource. This is so because the magnitude of the externality agent 2 exerts on agent 1 is much larger than the one of the externality agent 1 exerts on agent 2. The effect of an increase in the level of consumption of agent 2 is then larger in agent's 1 utility than the effect of an increase in his own level of consumption.

[8]Recall that we have normalized the total amount of resources to one. This is, of course, without lose of generality because of linearity.

When considering such kind of situations the implications on the solution to the Nash bargaining problem is immediate. Since the Nash bargaining solution has to be Pareto efficient and there is a unique Pareto efficient allocation, the Nash bargaining problem is trivially solved.

We will disregard the kind of economies that we have just described. Indeed, we will concentrate from now on in the completely opposite kind of situations. We will only consider economies where any allocation that exhausts available resources is Pareto efficient. When this happens we say that the economy is *regular*.[9] The assumption of a regular economy ensures there is a nontrivial bargaining problem and there exists competition among all agents to obtain some share of the unit of resources.

The next result provides a complete characterization of regular economies in terms of the matrix of network externalities. Before stating it we define a useful notion for the analysis in the rest of the paper.

Definition *We say that an n-dimensional vector $\boldsymbol{\mu}$ is a strict system of weights if and only if $\mu_i > 0$ for every $i \in \mathcal{N}$ and $\sum_{i=1}^{n} \mu_i = 1$.*

Now we provide the first characterization result of regular economies.

Proposition 2 *An economy $\mathbf{B}$ is regular if and only if there exists a unique strict system of weights $\boldsymbol{\mu}$ and a positive constant $\kappa > 0$ such that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)}(\mathbf{B}) = \kappa$ for every $i \in \mathcal{N}$.*

The previous result characterizes regularity through the matrix of network externalities. Each element of column $\mathbf{e}^{(i)}$ expresses how large is the network externality agent i exerts on each agent. When we compute the weighted average of these elements we obtain a single value that expresses an overall measure of network externalities exerted by agent i. Hence, Proposition 2 says that regularity amounts to find normalized weights, which are unique and depend on the economy we are analyzing, such that this overall measure of network externalities each agent exerts is positive and equal for all agents.

It is also possible to provide a characterization of regular economies in terms of the primitives of the economy, i.e. the matrix $\mathbf{B}$ of bilateral influences.

Proposition 3 *A necessary and sufficient condition for an economy $\mathbf{B}$ to be regular is that*

$$\sum_{j=1, j\neq i}^{n} b_{ji} < 1 \quad \textit{for every } i \in \mathcal{N}$$

[9] Observe that this is not the unique other possible situation. It could be that only a subset of agents should consume for an allocation to be Pareto efficient. For a generic characterization of all possible situations in terms of endogenous centrality measures derived from the position of each agent in the network of bilateral influences, see de Martí Beltran (2006).

This result provides a simple and direct way to check if an economy is regular, and it helps us to understand better which are the network forces that induce regularity. It states that regularity amounts to requiring that the aggregate level of bilateral influences each agent exerts on others, $\sum_{j=1,j\neq i}^{n} b_{ji}$, is not too large, in fact not larger than 1.

For example, consider an economy where all agents are connected to each other and the level of bilateral influence across any pair of individuals is equal to certain value α. In this case the necessary and sufficient condition for such kind of economy to be regular is that $\alpha < \frac{1}{n-1}$. this condition is satisfied if $\alpha < 0$, and only if bilateral influences are positive the condition bounds α. This is quite natural. When influences exerted are negative, distributional conflict naturally arises because each agent wants the rest of the economy to receive the smallest possible share. Any allocation that exhausts resources is in this case Pareto efficient. Instead, when influences are positive and sufficiently large each agent would prefer that other agents receive all the resource since this would increase more his utility than receiving himself part of it, and the Pareto frontier might degenerate to a single point.

Consider a regular economy $\mathbf{B}$. Then we can compute easily its associated strict system of weights $\boldsymbol{\mu}$ and constant κ from Proposition 2.[10]

Lemma 1 *Let* $\mathbf{B}$ *be a regular economy. Define* $\delta_i = 1 - \sum_{j=1,j\neq i}^{n} b_{ji}$*,* $i \in \mathcal{N}$*, and* $\delta = \sum_{i=1}^{n} \delta_i$*. Then,* $\kappa = \frac{1}{\delta}$ *and* $\mu_i = \kappa\delta_i$ *for all* $i \in \mathcal{N}$*.*

The value of δ_i is negatively related to the level of aggregate bilateral influences agent i exerts. Hence, μ_i is smaller the larger this aggregate level of bilateral influences is. The next section studies more in detail how these constants relate to the particular network structure of influences, and we provide interpretations for them.

3.2 Efficiency and Network Centrality

We can provide an alternative interpretation to the efficiency characterization in proposition 2. To this end, we have to introduce some terminology derived from the literature on social networks.

Given a network we can try to measure the prominence of each agent due to her position in the network. There are several variables that can determine the prominence of an actor in a network. Furthermore, the definition of prominence may depend on the setting we are studying. It is not the same if we deal with directed or undirected networks, or with weighted or unweighted networks. Hence, there is not in the social network analysis literature a unique standard definition of prominence.

The more usual concept to analyze prominence in networks is centrality. It is fairly natural to associate prominence with connectivity and this is what centrality measures do. In the case of weighted and directed networks, as the ones we are considering in our analysis, a rough measure

[10] The proof of Lemma 1 is contained in the proof of Proposition 3.

of centrality of agent i would be the sum of weights of the links that point to agent i, $S_i = \sum_{j \neq i} b_{ji}$.[11] This measure is called the degree centrality of agent i. In terms of our structural influence model, S_i measures the aggregate level of influence that emanates from agent i.

Note that $\delta_i = 1 - S_i$, which is a positive quantity whenever the economy $\mathbf{B}$ is regular, is then a complementary degree centrality index for agent i. Its value is smaller the largest is the degree centrality measure S_i. Therefore, μ_i is also a complementary centrality index, since it is a renormalization of δ_i to make the sum of the indices for all agents to add up to one.

While this degree centrality is informative of some kind of prominence derived by the way influences vary across pairs, it does not capture the value of how these influences spread indirectly along chains of bilateral influences.

Remember that, as we have explained before, given an economy $\mathbf{B}$ we have that for any $l \geq 1$ the matrix $\mathbf{B}^l$ keeps track of the l-order network externalities: each entry $b_{ij}^{[l]}$ equals the sum of weights of all paths of length l from i to j. Hence, to construct a more elaborate centrality measure we might include these indirect network effects subsumed in the sequence of matrices $\left\{\mathbf{B}^l\right\}_{l \geq 1}$. A natural way is to consider a decay factor $\lambda \in (0, 1]$ and weight the l-order network effects by λ^l. This is the Katz-Bonacich centrality measure. The (unweighted) Katz-Bonacich inner-centrality[12] measure vector, $\mathbf{b}(\mathbf{G}; \lambda)$ is defined as:

$$\boldsymbol{\kappa}(\mathbf{B}; \lambda) = \left(\sum_{l=0}^{\infty} \lambda^l \mathbf{B}^l \right)^t \cdot \mathbf{1}$$

Whenever this vector is well-defined we can rewrite it as:

$$\boldsymbol{\kappa}(\mathbf{B}; \lambda) = \left[(\mathbf{I} - \lambda \mathbf{B})^{-1} \right]^t \cdot \mathbf{1}$$

A variation of this measure, called the weighted Katz-Bonacich centrality measure, is the following.

Let $\boldsymbol{\mu}$ be an strict system of weights. Then the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure, $\kappa_{\boldsymbol{\mu}}(\mathbf{B}; \lambda)$, is given by the following formula:

$$\boldsymbol{\kappa}_{\boldsymbol{\mu}}(\mathbf{B}; \lambda) = \left[(\mathbf{I} - \lambda \mathbf{B})^{-1} \right]^t \cdot \mu$$

In the unweighted Katz-Bonacich centrality measure all agents count the same when considering the sum of network effects generated by each one of them. In the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure the network effects generated by agent i are counted with weight μ_i. Some agents count more than others when aggregating the whole matrix of network effects $\mathbf{E}(\mathbf{B})$.

[11] This is an inner-centrality measure. Alternatively, we could define an outer-centrality measure by the sum of weights of the links that start in agent i. Since, as we will show in a moment, in our analysis the relevant centrality measure is an inner measure, we avoid this possible distinction in the text.

[12] It is an inner measure of centrality because it measures weights of paths and cycles that end on each agent. An outer-centrality measure could be defined without transposing in the following equation.

After this digression into the realm of social and economic networks, we can reinterpret the condition of proposition 3 making use of weighted Katz-Bonacich centrality measures. The condition is equivalent to say that there exists a unique strict system of weights μ such that

$$\left[(\mathbf{I}-\mathbf{B})^{-1}\right]^{t}\cdot\boldsymbol{\mu}=\kappa\mathbf{1}$$

with κ being a positive constant. The reader can immediately recognize the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure, with $\lambda = 1$, in the left handside of the last equation. Hence the regularity condition says that there exists a vector of weights for which the weighted Katz-Bonacich centrality measure is equal, and positive, for all agents. This individual index measures the aggregate level of network influence effects that i generates. These are represented by the paths on the network that finish on i, and this is exactly what the Katz-Bonacich centrality index takes into account.

Two comments are in order. First, observe that our model generates endogenously the unique system of weights $\boldsymbol{\mu}$ for which this centrality condition is satisfied. This is Proposition 3. Second, the decay factor is equal to 1, and hence direct and indirect influences count the same to compute this measure of prominence. Hence, we can rewrite proposition 2 as follows

Proposition 2' *The economy* $\mathbf{B}$ *is regular if and only if there exists a unique strict system of weights* $\boldsymbol{\mu}$ *and a constant* $\kappa > 0$ *such that*

$$\boldsymbol{\kappa}^{i}_{\boldsymbol{\mu}}(\mathbf{G};1)=\kappa \qquad for\ all\ i\in\mathcal{N}$$

A general characterization, not only for regular economies, of Pareto efficiency in economies with pairwise influences by means of centrality measures can be found in de Martí (2006).

4 Bargaining and Influences

4.1 The Bargaining Problem and its Solution

From now on, we consider only regular economies with influences. We turn to the study of distributional conflict for these economies.

We consider the classical and widely used Nash bargaining solution (Nash, 1950). Following this seminal work we define an n-person bargaining problem as a duple $\langle X, \mathbf{d}\rangle$, where $X \subset \mathbb{R}^n$ is a convex and compact set that expresses the utility possibility set in the economy, and $\mathbf{d} \in X$ is the disagreement point, that expresses the utilities each agent would obtain in case they are not able to reach an agreement. The disagreement point has to satisfy the following dominance condition: there exists $\mathbf{v} \in X$ such that $\mathbf{v}$ strictly Pareto dominates $\mathbf{d}$, i.e. $v_i > d_i$ for every $i \in \mathcal{N}$. The (symmetric)[13] Nash bargaining solution $\mathbf{x}^S = \left(x_1^S, \ldots, x_n^S\right)$ to $\langle X, \mathbf{d}\rangle$ is the solution to the following maximization problem

$$\max_{\mathbf{x}\in X} \prod_{i=1}^{n} (x_i - d_i)$$

Due to convexity of the utility possibility set X and strict convexity of the objective function this problem has a unique solution.

We want to analyze this Nash bargaining solution in the case the utility possibility set X is induced from a regular economy with influences. Observe this is possible since as we mentioned before UPS $(\mathbf{B})$ is convex and compact for any economy $\mathbf{B}$.

Given an economy $\mathbf{B}$, let $\mathbf{u}^{min} = \left(u_1^{min}, \ldots, u_n^{min}\right)$ be the utility vector where each entry u_i^{min} is equal to the minimal utility agent i can obtain within the set of efficient allocations of economy $\mathbf{B}$.[14] Since we assume that the economy is regular, from Proposition 2 we know that there exist only one strict system of weights $\boldsymbol{\mu}$ and one positive constant κ such that $\boldsymbol{\mu}\cdot\mathbf{e}^{(i)} = \kappa$ for every $i \in \mathcal{N}$. Given a disagreement point $\mathbf{d}$ we relabel agents from 1 to n such that $\mu_1\left(u^{min} - d_1\right) \geq \cdots \geq \mu_n\left(u_n^{min} - d_n\right)$. Finally, let

$$\psi^{(0)} = \frac{1}{n}(\kappa - \boldsymbol{\mu}\cdot\mathbf{d})$$

and let

$$\psi^{(j)} = \frac{1}{n-j}\left(\kappa - \boldsymbol{\mu}\cdot\mathbf{d} - \sum_{k=1}^{j} \mu_l\left(u_k^{min} - d_k\right)\right)$$

for any $j \in \{1, \ldots, n-1\}$.

Now we have all the necessary ingredients to characterize the Nash bargaining solution for any regular economy with influences. This is done in the following result.

Proposition 4 *Consider a regular economy $\mathbf{B}$. Then, there exists $j \in \{0, \ldots, n-1\}$ such that the utility vector associated with the Nash bargaining solution, $\mathbf{u}^S$, is*

$$u_i^S = u_i^{min} \qquad if\ i \leq j$$

and

$$u_i^S = d_i + \psi^{(j)}\frac{1}{\mu_i} \qquad if\ i > j$$

The Nash bargaining solution allocation is equal to $\mathbf{c}^S = (\mathbf{I} - \mathbf{B})\,\mathbf{u}^S$.

This result characterizes the utilities and levels of consumption of the Nash bargaining solution for any regular economy. In particular, it characterizes corner, partially corner and interior solutions.

[13] To simplify the analysis we only consider the symmetric Nash bargaining solution. The analysis for asymmetric Nash bargaining solutions with heterogeneous bargaining power is completely analogous.

[14] In fact, $\mathbf{u}_i^{min} = \min\{e_{i1}(\mathbf{B}), \ldots, e_{in}(\mathbf{B})\}$, so this vector of minimal utilities can easily be derived from the group influence matrix.

For an individual i that obtains positive gains, $u_i^S - d_i > 0$, these gains are inversely proportional to μ_i. If both i and j obtain positive gains we have that

$$\frac{u_i^S - d_i}{u_j^S - d_j} = \frac{1 - \sum_{k \neq j} b_{kj}}{1 - \sum_{k \neq i} b_{ki}} \tag{6}$$

The relative gains of agent i with respect to those of agent j uniquely depend on the level of aggregate influence exerted by agent i and agent j. In particular, the largest is the magnitude of aggregate influence that emanates from agent i compared with those that emanate from agent j, the largest the relative gains of i with respect to j.

Aggregate influence levels determine relative gains for those agents that obtain positive gains. This does not mean that these levels form the unique relevant information from the structural influence model to characterize the Nash bargaining solution. The minimal utilities profile, $\mathbf{u}^{\min}$, and therefore the multiplier $\psi^{(j)}$, can not be expressed in terms of the aggregate influence levels. Indeed, minimal utilities internalize all levels of network influence effects, since u_i^{min} equals the minimal entry in i's row of matrix $\mathbf{E}(\mathbf{B})$. In non-interior solutions where some agents obtain no gains from bargaining the information from the matrix of network externalities is fundamental for the characterization of the Nash bargaining outcome.

The multiplier $\psi^{(j)}$ represents the remaining surplus, the remaining value of the available unit of resources in terms of utilities, once we substract the minimal utilities some of the agents obtain (agents $k \leq j$). The rest of this remaining value is shared proportionally to the inverse of entries of $\boldsymbol{\mu}$.

Example 3. The analysis of the following two economies illustrate the characterization we have just provided in a 2-agents setting. Economy (a) is such that $b_{12} = 4/5$ and $b_{21} = 1/4$ while economy (b) is such that $b_{12} = b_{21} = 1/2$. The matrices of network externalities for each economy are

$$\mathbf{E}_{(a)} = \begin{pmatrix} 5/4 & 1 \\ 5/16 & 5/4 \end{pmatrix} \qquad \mathbf{E}_{(b)} = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$$

The utility possibility set with the respective Nash bargaining solution depicted, when the disagreement point is $\mathbf{d} = \mathbf{0}$, in both cases are

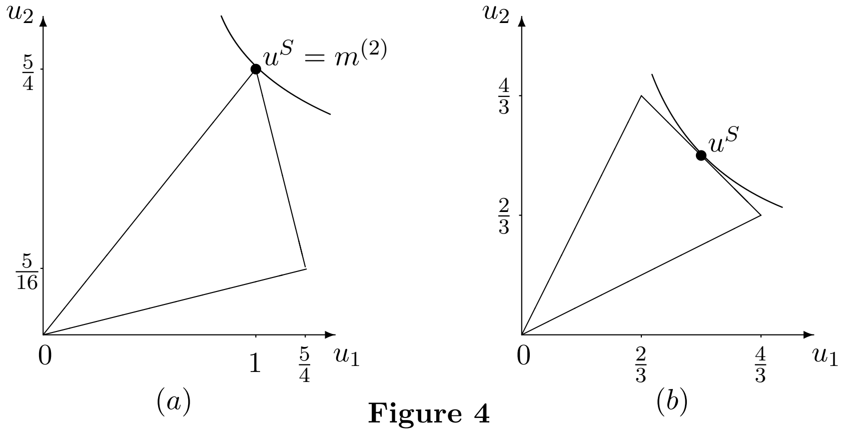


Figure 4

In example (a) we obtain a corner solution. One agent receives all the resource as the solution to the distributional conflict. We have that $\delta_1 = 3/4$, $\delta_2 = 1/5$, and therefore $\delta = 19/20$. The associated constant and strict system of weights from Proposition 2 are $\kappa = 20/19$ and $\mu_1 = 15/19$, $\mu_2 = 4/19$. The minimal utility each agent can obtain in a Pareto efficient allocation is $u_1^{min} = 1$ and $u_2^{min} = 5/16$. In this case the j from proposition 4 equals 1, and the multiplier is $\psi^{(1)} = 5/19$. In the solution, agent 1 receives nothing and agent 2 receives all the resource. Agent 1 obtains his minimal utility, $u_1^s = 1$, while agent 2, instead, obtains $u_2^S = \psi^{(1)}\frac{1}{\mu_2} = \frac{5}{19}\frac{19}{4} = \frac{5}{4}$, that equals his maximal possible utility within the set of efficient allocation.

In example (b) we obtain an interior solution. Both agents obtain a utility within their minimal and maximal utility. In this case, we have that $\delta_1 = \delta_2 = 1/2$,and therefore $\delta = 1$. The associated constant and strict system of weights from proposition 2 are therefore $\kappa = 1$ and $\mu_1 = 1/2$, $\mu_2 = 1/2$. The minimal utility each agent can obtain in an efficient situation is $u_1^{min} = u_2^{min} = 2/3$. We get that the j from proposition 4 equals 0 and the multiplier is $\psi^{(0)} = 1$. Hence, each agent obtains a utility equal to $u^s 1 = u_2^s = \psi^{(0)}\frac{1}{\mu_i} = 2$.

4.2 Discussion

4.2.1 A Geometric Characterization

We can provide a graphical approach of how the Nash bargaining solution is obtained in the case of an interior solution. A similar kind of interpretation can be given for corner and semi-corner solutions but then the analysis is more involved.

In proposition 4 we have obtained a complete characterization of the Nash bargaining solution, both in terms of utilities and shares received. In particular a fundamental ingredient for this characterization is the unique strict system of weights μ from proposition 2. Call $\boldsymbol{\mu}^{-1}$ the vector with entries the inverses of the entries of $\boldsymbol{\mu}$, i.e. $\mu_i^{-1} = 1/\mu_i$. If the Nash bargaining solution is interior, the vector of utilities agents obtain is equal to the disagreement point plus a positive multiple of vector μ^{-1}. From this construction we can derive a geometric procedure to deduce when the Nash bargaining solution is interior given a particular economy $\mathbf{B}$. We present it with the use of the two previous examples.

We depict again the utility possibility sets and the vectors μ and μ^{-1} for each example.[15]

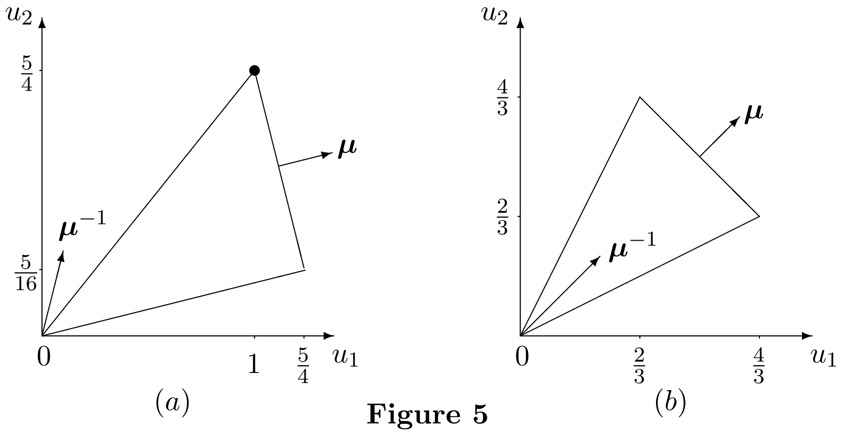


Figure 5

In a regular economy the vector $\boldsymbol{\mu}$ is orthogonal to the Pareto frontier in the utility possibility set.[16] We have depicted the vector $\boldsymbol{\mu}^{-1}$ in the point $(0, 0)$ because this is the disagreement point.

In the first example, since the vector $\boldsymbol{\mu}^{-1}$ lies outside the utility possibility set, no multiple can intercept the Pareto frontier. Therefore, the solution can not be interior.

In the second example, the vector $\boldsymbol{\mu}^{-1}$ lies inside the interior of the utility possibility set and then the Nash bargaining solution can be obtained by multiplying the vector by a positive scalar until it touches a point of the Pareto frontier. This point is the Nash solution utility vector.

Just to clarify that this geometric procedure is valid for any possible disagreement point, and not only for the case in which $\mathbf{d} = \mathbf{0}$, we show here how it applies also for the first economy in the

[15]We have rescaled both vectors for convenience. This does not affect at all the reasoning.

[16]The conditions on Proposition 2 determine a hyperplane with orthogonal vector $\boldsymbol{\mu}$. The utility possibility set is a subset of this hyperplane.

case that $\mathbf{d} = \mathbf{u}^{min}$.

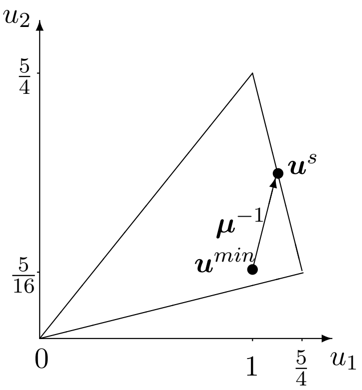


Figure 6

We situate the vector $\boldsymbol{\mu}^{-1}$, that does not change with the change of disagreement point since it only depends on the matrix $\mathbf{B}$, in the point $\mathbf{u}^{min} = (1, 5/16)$. The point where a positive rescaling of $\boldsymbol{\mu}^{-1}$ touches the Pareto frontier coincides with the Nash bargaining solution utilities profile, $\mathbf{u}^s$.

4.2.2 The Disagreement Point

As stressed in Binmore et al. (1996), the choice of a particular disagreement point entails also some of the features of the bargaining process when considering the Nash bargaining solution. In our case two different possible disagreement points emerge as the more natural choices.

The first one is the choice of $\mathbf{d} = \mathbf{0}$. This would be the natural choice when in the bargaining situation there are time concerns. In this case, agents now that an agreement could be infinitely delayed and therefore obtain no utility at all. This time concerns are modelled by means of the choice of the zero vector as the disagreement point.

Another possibility is to choose $\mathbf{d} = \mathbf{u}^{min}$, that is that d_i coincides with the minimal utility agent i can obtain in an efficient situation, u_i^{min}. In a regular economy, this vector of minimal utilities satisfies the property of domination that the disagreement point has to satisfy. Observe that this disagreement point derives endogenously from the pattern of influences expressed by matrix $\mathbf{B}$. It might be a natural selection when considering situations in which no time concerns exist. When Pareto efficiency is a requirement of the solution to the distributional conflict, as it is in the case of the Nash bargaining solution, agent i might make recognize the rest of members in the economy that he should not obtain less than u_i^{min}.

Hence, while our cooperative approach can not capture all of the features of particular applications, some of these features can be incorporated into the model, not by changing utilities but directly through the choice of the disagreement point.

Corollary 1 *The Nash bargaining solution is always interior when* $\mathbf{d} = \mathbf{u}^{\min}$. *The Nash bargaining solution is interior when* $\mathbf{d} = \mathbf{0}$ *if and only if for all* $i \in \mathcal{N}$

$$\sum_{j \neq i} b_{ij} \frac{\delta_i}{\delta_j} < 1 \tag{7}$$

This condition resembles the condition for regularity stated in Proposition 3. However, it is different in two aspects. First, the set of pairwise influences that appear are in this case the ones that i receives, instead of those that i exerts. Second, these bilateral influences are weighted by the quotient

$$\frac{\delta_i}{\delta_j} = \frac{1 - \sum_{k \neq i} b_{ki}}{1 - \sum_{k \neq i} b_{kj}}$$

For example, in the case that all bilateral influences are positive this quotient is larger than 1 if $\sum_{k \neq i} b_{kj} > \sum_{k \neq i} b_{ki}$. The influence j exerts on i is weighted by larger values in condition (7) if j exerts a larger aggregate level of direct influences on others than i. Observe that this quotient was also present when computing relative profits obtained from bargaining across pairs of individuals.

When we fix a disagreement point $\mathbf{d}$ such that for some agent $u_i^{min} > d_i$ we also impose a value to the minimal gains that this agent is going to obtain from the bargaining situation. This value is equal to $u_i^{min} - d_i$. It might be possible that these minimal gains from bargaining can not be reconciled with the conditions imposed on relative gains across individuals in (6) when agents obtain a positive share of the resource. The conditions in Corollary 1 exactly account for this fact, and provide the expressions that ensure that this tension does not arises.

To better understand that interiority condition when $\mathbf{d} = \mathbf{0}$, we analyze in more depth the case of two agent economies. Given an economy

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$$

the values of δ_1 and δ_2 are $\delta_1 = 1 - b_{21}$ and $\delta_2 = 1 - b_{21}$. Given the regularity condition, this two values are positive. The conditions for interiority expressed in the previous corollary are in this case,

$$\begin{aligned} \frac{1 - b_{12}}{1 - b_{21}} &> b_{12} \\ \frac{1 - b_{21}}{1 - b_{12}} &> b_{21} \end{aligned}$$

The following reasoning helps understand when we lose interiority. Fix a value $b_{21} < 1$. If $b_{12} = b_{21}$ the conditions reduce to $1 > b_{12}$ and $1 > b_{21}$ which are trivially satisfied because of regularity. When we increase b_{12} the second condition is still satisfied since the left-hand side increases. But the left-hand side of the first condition increases while the right hand side of this same condition increases. If we increase b_{12} enough it is possible that this first condition is not satisfied for the parameters. It becomes too difficult to control for both conditions. Agent 2 exerts a larger influence on agent 1 than the influence agent 1 exerts on agent 2. If this difference is sufficiently asymmetric, and this asymmetry is measured by the two conditions above, then one of the agents receives all the resource, even if the economy is regular.

4.3 Nonparticipants and the Bargaining Outcome

Until now, we have considered that all agents in the economy are involved in the bargaining problem. However, our model also provides a framework to study what would happen if some agents in the economy do not participate in the bargaining problem but care for some of the agents that indeed participate.

The following example shows in a simple economy how we can use the tools we have developed so far to clarify the effect of agents that do not participate in the bargaining problem. It is very close to the example developed in Kalai (1977).

Example 4. Consider the economy with three agents represented by the following network

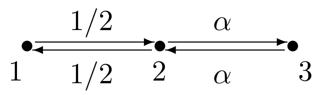


Figure 7

where α is a positive constant. Suppose only agents 1 and 2 are engaged in a bargaining problem. How does the introduction of agent 3 into the model[17] perturbs the bargaining outcome respect to the situation when we consider agent 1 and 2 in isolation? The solution to the bargaining problem without the presence of agent 3 is $\mathbf{c}^S = (1/2, 1/2)$, as we have previously described. If we take care of the existence of agent 3 we could proceed as follows.

The matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}$$

Hence, the matrix of network externalities when considering the 3 agents is

$$\mathbf{E}(\mathbf{B}) = \frac{4}{3 - 4\alpha^2} \begin{pmatrix} 1 - \alpha^2 & 1/2 & \alpha/2 \\ 1/2 & 1 & \alpha \\ \alpha/2 & \alpha & 3/4 \end{pmatrix}$$

[17]In the social preferences interpretation we can interpret agent 3 as a relative of agent 2.

Each entry e_{ij} of this matrix represents the network externality magnitude agent j exerts on agent i if we consider the pattern of bilateral influences within all agents in the economy, included agent 3. Now, we could solve for the bargaining problem as we did previously in this section considering the submatrix $\mathbf{E}_{1,2}$ obtained eliminating from $\mathbf{E}(\mathbf{B})$ the third row and column

$$\mathbf{E}_{1,2}(\mathbf{B}) = \frac{4}{3-4\alpha^2} \begin{pmatrix} 1-\alpha^2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

to characterize the Nash bargaining solution for the bargaining problem that involves only agent 1 and 2, but taking care of the initial levels of interdependence of the three agents in the economy. In particular the allocation that solves this problem is

$$\mathbf{c}^S = \left(\frac{1}{2} \cdot \frac{1-4\alpha^2}{1-2\alpha^2}, \frac{1}{2} \cdot \frac{1-\alpha^2}{1-2\alpha^2} \right)$$

Observe the change in the pattern of consumption. If α is positive and sufficiently small then $c_1^S < 1/2$ and $c_2^S > 1/2$. Bilateral influences agents 2 and 3 exert on each other alters the bargaining power of agent 2 with respect to agent 1, and agent 2 obtains a larger share of the budget.

This methodology can be generalized to any number of agents. If the economy is formed by $n_1 + n_2$ agents and only the first n_1 of them are engaged into a bargaining problem, first we have to compute the matrix of network externalities $\mathbf{E}(\mathbf{B})$ for the economy as a whole, i.e. considering all the $n_1 + n_2$ agents. Then, we solve the bargaining problem using the submatrix formed by the first n_1 rows and columns, $\mathbf{E}_1(\mathbf{B})$, as if this last matrix was the matrix of network externalities of this (sub-)economy. In this way we internalize all the network effects generated by the structural influence pattern into the bargaining problem played by the first n_1 individuals, when considering all agents in the economy.

The next proposition provides a characterization of the matrix $\mathbf{E}_1(\mathbf{B})$ for such a situation. Let $\mathbf{B}_{11}$ be the matrix of bilateral influences across participants, $\mathbf{B}_{22}$ be the matrix of bilateral influences across nonparticipants, $\mathbf{B}_{12}$ be the matrix of bilateral influences from nonparticipants to participants, and $\mathbf{B}_{21}$ be the matrix of bilateral influences from participants to nonparticipants.

Proposition 5 *If $n = n_1 + n_2$, with $1 < n_1 < n$ being the number of members of the economy that participate in the bargaining game, then*

$$\mathbf{E}_1 = \left(\mathbf{I}_{n_1} - \mathbf{B}_{11} - \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21} \right)^{-1}$$

This matrix has a natural interpretation. Observe that it is equivalent to the matrix of network externalities that we would obtain if the economy had only the n_1 agents that are involved in the distributive conflict and the matrix of bilateral influences were $\mathbf{B}_{11} + \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21}$. Matrix $(\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1}$ equals the matrix of network externalities if only nonparticipants were in the economy, $\mathbf{E}(\mathbf{B}_{22})$. Hence, this new associated matrix of bilateral influences across participants accounts on the feedback effect, derived from influence exerted by participants on nonparticipants and viceversa, of network externalities exerted within the subeconomy formed by nonparticipants. From right to left the matrix

$$\mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21}$$

is obtained by first taking into account the direct bilateral influences participants exert on non-participants, then network effects among non-participants are computed, and finally we compute how these come back again to partipants through direct influence exerted by non-participants on participants. the product of these three effects is added to the initial matrix of bilateral influences across participants to compute the solution to the Nash bargaining problem.

In our framework, if there were no influences, i.e. $b_{ij} = 0$ for all pairs ij, then the bargaining power of each agent would exactly coincide with the Nash bargaining solution share he receives. Once influences are introduced, any change in the levels of consumption of the Nash bargaining solution shares profile can be interpreted as a change in bargaining power due to the position in the network of bilateral influences. Here, network externalities, the aggregation of direct and indirect network effects, generates the pattern of bargaining power, if we want to interpret the allocation solution as the solution of an asymmetric bargaining problem without influences.

Kalai (1977) studies how non-participants in a Nash bargaining problem induce a change of bargaining power across participants, interpreting each non-participant as a replica of the participant for which this agent cares. In our case, each non-participant can care at the same time for different participants with varying intensities and we can provide the particular mapping from this interdependency structure to bargaining asymmetric outcomes.

4.4 α-economies.

In this section we study a family of networks with some particular properties.

Let $\alpha \in \mathbb{R}_+$. We say that an economy is an α-economy if whenever $b_{ij} \neq 0$, then $b_{ij} = b_{ji} = \alpha$. Hence, in an α-economy whenever there is a bilateral influence this influence is bidirectional and of same weight.[18]

In this family of economies the heterogeneity comes only from one source, the network geometry, and not from heterogeneous influence levels across pairs of agents. Therefore the analysis of this family of networks sheds some light on the isolated effect of the network geometry on the Nash bargaining outcome. In fact, as we show in the following lines, the characterization of the Nash bargaining solution becomes very transparent under some mild assumptions.

Fixed α and an α-economy $\mathbf{B}$, we define the degree of agent i, that we denote by $deg_i(\mathbf{B})$, as

$$deg_i(\mathbf{B}) = \frac{1}{\alpha} \sum_{j \neq i} b_{ij} \tag{8}$$

[18] Observe that in the family of α-economies $\mathbf{G} = \mathbf{B}$, since matrix $\mathbf{B}$ is symmetric.

The degree of an agent is a measure of connectivity. It equals the number of connections an agent has in the network of bilateral influences. Due to the symmetric nature of α-economies, the degree of an agent computes at the same time to how many people this agents exerts a direct influence, and from how many people this agent receives a direct influence.

Suppose that if agents do not agree in a division of the resource the disagreement outcome is that no division is implemented and hence agents receive a utility equal to 0, i.e. $\mathbf{d} = \mathbf{0}$. Then, under the regularity condition $1 - (n-1)\alpha > 0$ that ensures that for a fixed α any α-economy is regular, we obtain the following characterization:

Proposition 6 *Let* $\mathbf{B}$ *be a regular* α*-economy. Then the Nash bargaining solution is interior and the utility each agent obtains is*

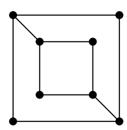
$$u_i^S = \frac{1}{n\left(1 - \alpha deg_i\left(\mathbf{B}\right)\right)}$$

Hence,

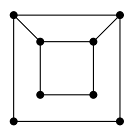
$$c_i^S = \frac{1}{n\left(1 - \alpha deg_i\left(\mathbf{B}\right)\right)} - \sum_{j \neq i} b_{ij} \frac{1}{n\left(1 - \alpha deg_j\left(\mathbf{B}\right)\right)}$$

Hence, for α-economies the degree of an agent is the unique element relevant from the network to determine the utility this agent obtains. Observe that the share an agent obtains does not only depend on its own degree but also on the degrees of agents to which he is connected.

Example 5. Consider the following two networks.



$\mathbf{B}_a$



$\mathbf{B}_b$

In both networks there are four agents with three neighbours and four with two neighbours. However, if for example $\alpha = 0.1$, agents with three neighbours agents with three neighbours receive a larger share of the pie than in the second network, while the opposite applies for agents with two links. The following table provides the shares in both cases for the members of each class.

Nash Shares with $\alpha = 0.1$	$\mathbf{B}_a$	$\mathbf{B}_b$
Agents with 3 links	0.129	0.127
Agents with 2 links	0.121	0.123

This proves that the degree distribution is not a sufficient invariant to determine how the resource is distributed, it is also important the particular geometry of how agents are connected. This is expressed in the following corollary.

Corollary 2 *The share an agent obtains in the Nash bargaining solution increases with the number of neighbours he has and diminishes with the number of neighbours that his neighbours have.*

That is why we obtain different shares in the previous pair of networks. In the first one each agent that has three neighbours has one neighbour with three links and two with two links, while in the right one each agent with three links has two neighbours with three links and one with two. In accordance with this last corollary, this agents should obtain a smaller share in the network at the right, that is what we have observed before.

This last corollary provides in fact the main intuition on how the Nash bargaining solution internalizes influences. The share an agent receives depends on the aggregate level of bilateral influences provided at the local level. The more connected an agent is and the less connected his neighbours are, the more valuable is the share this agent receives for the spread of influence through the network. If, instead, his neighbours are also very connected, it is not necessary to give more to this agent. In this case other agents receive larger shares than before because they help more to spread the effect of influences all over the economy. The Nash bargaining solution with influences internalizes this indirect effects.

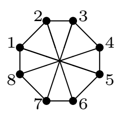
A particular case are degree-regular α-economies. In such economies all agents have the same degree. Here pattern does not matter. In fact, we have the following corollary.

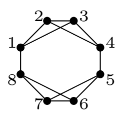
Corollary 3 *Let $k \in \{0, \ldots, n-1\}$ and let $\mathbf{B}$ be a regular α-economy such that $deg_i(\mathbf{B}) = k$ for all $i \in \mathcal{N}$. Then all agents obtain the same utility and consume the same quantity in the Nash bargaining solution. In particular $c_i^S = 1/n$ for all $i \in \mathcal{N}$ and*

$$u_i^S = \frac{1}{n(1-k\alpha)} \qquad \forall i \in \mathcal{N}$$

Therefore in the case of regular -economies it does not matter how agents are connected but how many connections each agent has.

Example 6. The following two networks, with each link being bidirectional and with same weight α, differ in their particular geometry and lead to the same solution to the bargaining problem.





The network depicted in the right side has larger clustering[19] levels than the one in the left. This is not an issue in the determination of the Nash bargaining solution. This example shows that clustering plays no role in the resulting division that solves the distributional conflict under degree-regularity.

[19]Clustering measures if the agents to which an agent is connected are also connected within them.

5 The Effect of Network Changes on Welfare and Consumption

In this section we explore how changes in the network of bilateral influences[20] translate into changes in welfare and consumption patterns. In particular, we center our attention in how a differential change on the weight of a link changes the utility and the level of consumption in equilibrium of each agent involved in the bargaining game. Hence, our results help to understand how changes in the magnitude of influences change the characteristics of the bargaining outcome.

For the sake of simplicity, we focus our attention in situations where the bargaining solution is interior, meaning that $c_i^S > 0$ and, hence, $u_i^S > u_i^{min}$ for every $i \in \mathcal{N}$.[21]

The following proposition provides conclusions on comparative statics related to the utility pattern of the Nash bargaining solution.[22]

Proposition 7 *Let* $\mathbf{B}$ *define an economy with influences such that the Nash bargaining solution is interior. Then:*

(i) $\frac{\partial u_i^S}{\partial b_{kl}} \geq 0$ *if* $l \neq i$*, with equality if and only if* $d_l = 0$

(ii) sign $\left(\frac{\partial u_i^S}{\partial b_{ki}}\right) =$ *sign* $\left(1 - \sum_{j \neq i} \delta_j d_j\right)$ *for* $k \neq i$.

The first part of the proposition states that the Nash bargaining solution utility of agent i generally increases when there is an increase on the magnitude of a bilateral influence across any two other agents, whoever these are. Increases on bilateral influences agents different than i exert on each other are beneficial for agent i.

The second part of the proposition is a little bit more complex. It states that the increase on bilateral influences exerted by agent i are good for agent i if the term $\sum_{j \neq i} \delta_j d_j$ is sufficiently small (in fact, if it is smaller than one). Observe that this can happen either because the disagreement levels of agents different than i are small or because the levels of the δ's are small for agents different than i. Agent j has a small level of δ_j whenever he exerts on the aggregate large positive bilateral influences on other agents. Hence, for agent i it is good to exert larger positive influences, if other agents exert large aggregate levels of bilateral influences. The intuition is that larger bilateral influences exerted by agent i can increase indirect network influences from i to himself (through cycles in the network of bilateral influences) if other agents exert sufficiently high bilateral influences as well. If not, and for example other agents exert some bilateral negative influences, the indirect

[20] We make no distinction in defining the model in terms of networks or in terms of utilities. We refer to network changes because at some points this simplifies the necessary terminology.

[21] A more extensive analysis could be done to deal with corner solutions.

[22] If an economy is such that the Nash bargaining solution is interior, sufficiently small changes in the parameters of bilateral influences maintain interiority because of continuity. Hence, a comparative statics analysis in our setup is legitimated.

network effects can be negative for agent i and imply a decrease on utility.

We move now to comparative statics results related to consumption patterns. These are provided in the following result.

Proposition 8 *Let* $\boldsymbol{B}$ *define an economy with influences such that the Nash bargaining solution is interior. If* $\mathbf{d} = \mathbf{0}$,[23] *then:*

(i) $\frac{\partial c_i^S}{\partial b_{ki}} > 0$ *if* $k \neq i$

(ii) $sign\left(\frac{\partial c_i^S}{\partial b_{kl}}\right) = -sign\,(b_{il})$ *if* $k \neq i \neq l \neq k$

(iii) $\frac{\partial c_i^S}{\partial b_{ij}} > 0 \Leftrightarrow b_{ij} < -\delta_j$ *for* $j \neq i$

The first part of the proposition states a very simple and natural conclusion: agent i receives a larger share whenever the level of aggregate bilateral influences he exerts increases.[24] This generates a positive effect on several other agents in the economy through the spread of bilateral influences through network effects. We could also interpret this result in terms of bargaining power: since he exerts a larger aggregate level of bilateral influences his bargaining power increases, and that is why he gets a larger fraction of the resource.

The second part of the proposition states that if agent i receives a positive (resp. negative) direct bilateral influence from agent l then an increase of the direct bilateral influence agent l exerts on another agent k, different than i and l, implies that agent's i share diminishes (resp.increases). This is reminiscent of the result of the first part of the proposition, and hence the same kind of intuition applies.

Finally, the third part expresses that an increase in the weight of the direct bilateral influence agent l exerts on i implies an increase in agent i share if and only if the initial weight of this bilateral externality was sufficiently negative.

We recover example 3.(b) of section 4 to illustrate graphically how small changes on the levels of bilateral influences translate into changes on the utility and shares derived from the Nash bargaining solution. This example is the 2-person economy such that $b_{12} = b_{21} = 1/2$. If we increase b_{12} from the initial $b_{12} = 1/2$ to $\tilde{b}_{12} = 1/2 + \epsilon$, where $\epsilon < 1/2$ to satisfy the regularity condition, the new matrix of bilateral influences is

$$\tilde{\mathbf{B}} = \begin{pmatrix} 0 & 1/2+\epsilon \\ 1/2 & 0 \end{pmatrix}$$

[23] We consider this case since it is the more tractable one. In the proof the interested reader can find the exact expression of each one of the derivatives, no matter which disagreement point we consider.

[24] And the rest of bilateral influences do not vary.

and the new matrix of network externalities is

$$\tilde{\mathbf{E}}(\mathbf{B}) = \frac{4}{3-2\epsilon}\begin{pmatrix} 1 & 1/2+\epsilon \\ 1/2 & 1 \end{pmatrix}$$

The utility possibility set and the Nash bargaining solution behave as follows (dashed lines represent the initial situation and continuous lines represent the new one; u^S is the initial Nash bargaining solution and $\tilde{u}^S$ is the new one)

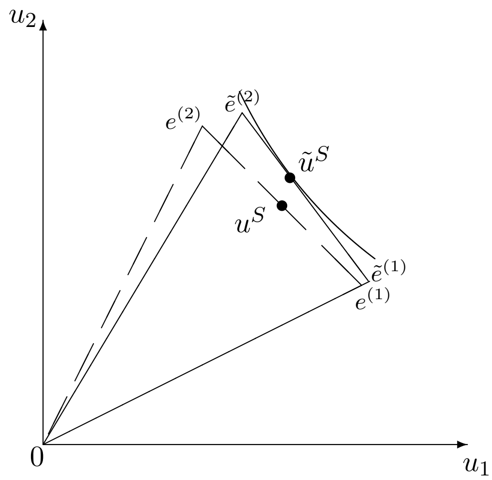


Figure 8

The increase on the level of altruism of agent 1 shifts the Pareto frontier upwards. Both agents obtain a larger utility in the new equilibrium of the bargaining game. This is consistent with the conclusions of proposition 7,[25] and now the division is no more half of the budget for each one. Indeed, the equilibrium point $\tilde{u}^S$ is closer to one of the extremes of the simplex than to the other. In particular it is closer to $\mathbf{e}^{(2)}$ which implies that c_2^S has increased while c_1^S has decreased. The increase of c_2^S is consistent with part (i) in proposition 8 while the decrease of $\mathbf{e}^{(1)}$ is consistent with part (iii), since $b_{12} = 1/2 > -1/4 = -\delta_2/2$.

6 Discussion

Until this point we have not discussed the suitability of the Nash bargaining solution in our setup. An initial point of debate is that with the use of this solution we abstract from the effect of coalitions when there are more than two agents. It could be possible that some agents decide to break relations with the rest of agents in the economy, and that this threat plays a role in the final

[25]Observe that the increase in u_1^S is the conclusion of part (i) while the increase of u_2^S is the conclusion of part (ii), since $\mathbf{d} = \mathbf{0}$.

division of resources. However, we do not think that in our setup this question plays a prominent role in the examples we have used to motivate our research.

Consider for example the example on urban crime. Except for the case in which the fear on crime is overwhelming, and citizens view the rest of problems associated with everyday urban life as secondary, it is difficult to imagine that this dispute on resources to fight against crime will lead to the division of the city. [26]

Similarly, in the social preferences example, we would not expect that members of a family would decide to break relations when natural daily distributional conflicts within the family arise.[27]

The omission of coalitions role is more controversial in the case of government spending. If only the simple majority of members of the government have to agree on the division of the budget, as it is the case in the analysis provided in Baron and Ferejohn (1989), this has non-negligible strategic implications on the final agreement reached. It has been our aim in this work to analyze in detail how the effects of pairwise influences affects distributional conflict. Of course, by adding to this pairwise influences pattern more institutional details, such as the majority voting rule in the case of government spending, we would obtain more accurate predictions of each particular example. [28]

Another consideration is the harmonization of the axioms that characterize the Nash bargaining solution and our setup with interdependent utilities. We provide here a discussion on this in terms of the alternative set of axioms proposed by Lensberg (1988). In particular, Lensberg shows that the Nash bargaining solution is the unique solution that satisfies *Pareto Efficiency*, *Anonimity* (if a utility possibility set is symmetric the solution is also symmetric) [29], *Scale Invariance* (when applying a linear transformation of utilities the solution changes accordingly to this linear transformation), and *Consistency* (if a subset of agents receive the utility they would receive in the solution, when applying the solution to the rest of the economy, the result is the same as if at the beginning we applied the solution to the economy as whole). [30]

Both *Pareto Efficiency* and *Anonymity* seem to be desirable properties of a bargaining solution. In our setup *Scale Invariance* is also desirable because when applying a linear transformation of utilities the preferences represented remain unperturbed.[31] Therefore, the axiom of *Scale Invariance* imposes that these equivalent utility representations lead to the same solution. Hence, if we consider the three previous axioms as natural requirements of a solution, the unique axiom for the

[26] Undoubtedly, there are cities in which this problem is real, and some neighbourhoods are introducing physical barriers to combat crime at private expenses. Is in these kind of situations in which our model would certainly not apply.

[27] But maybe it can be the case when dealing with a bequest.

[28] See Duggan(2004) for conditions about existence of equilibria in the n agents version of the Baron and Ferejohn game with externalities.

[29] More precisely, $if\ x \in UPS\ then\ \sigma(x) \in UPS\ for\ any\ permutation\ \sigma\ of\ the\ entries\ of\ x.$

[30] Nashs (1950) characterization substitutes Consistency by a probably more difficult to interpret axiom named Independence of Irrelevant Alternatives.

[31] This is because utilities in our model are additively separable with respect to consumption levels.

Nash bargaining solution that might deserve discussion is *Consistency*.

7 Conclusion

We have explored the outcome of the Nash bargaining problem with considering a simple model of interdependent behavior. Even if an economy is characterized by $n(n-1)$ variables, the model is tractable and we have been able to provide closed-form expressions for the bargaining outcome and comparative statics results. The network interpretation of the problem is helpful since it provides us with a set of tools that simplify the analysis and makes it more intuitive. It helps to understand the effect of heterogeneities in the model in all its dimensions, magnitude and pattern.

Part of the analysis in our work shows some similarities with previous work done by Kalai (1977).[32] Kalai interprets any agent that cares for a player in the bargaining problem but that do not participate in the bargaining problem, as a replica of this player. In our model, an agent can care for different players of the game, where this concern translates into influences as in the social preferences example in the introduction. The transmission of this concern is not done as a replication and its consequent change into the bargaining problem but through a pattern of different influences that affect players' behavior. In this sense, we allow for a more general pattern of interrelations and the transition is not done in a discrete manner, as replicas would do, but smoothly, since small changes in bilateral externality levels imply small changes in the levels of network effects.

Finally, our analysis borrows directly from the Nash bargaining solution. Different possible directions for further research are open. One possible direction could be to explore whether other cooperative solutions can be defined through some proper axioms adequate in a setting with heterogeneous influences such as the one developed in this work. Another possible direction is to go further in the study of non-cooperative bargaining models with an underlying structural pattern of bilateral influences. In particular, it might be valuable to study how the pattern of influences maps into equilibria of non-cooperative bargaining games that incorporate relevant features of particular applications, such as the voting rule in legislative bargaining, and how equilibria vary with respect to the case without influences.

[32] See also Lensberg and Thomson (1989) for some other work done with replicated agents in cooperative bargaining.

8 Proofs

Proof of Proposition 1

The determinant of the matrix $\mathbf{I}-\mathbf{B}$ is a polynomial in $n(n-1)$ variables. The set of points of $\mathbb{R}^{n(n-1)}$ in which this polynomial vanishes forms an algebraic variety of dimension $n(n-1)-1$ at most, and hence it is a set with Lebesgue measure equal to zero in $\mathbb{R}^{n(n-1)}$. ■

Proof of Proposition 2

The following lemma is useful.

Lemma 2 *Given a regular economy* $\mathbf{B}$*, a feasible allocation* $\mathbf{c}$ *is Pareto efficient if and only if there exists a strict system of weights* $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu}\cdot\mathbf{u}(\mathbf{c})\geq\boldsymbol{\mu}\cdot\bar{\mathbf{u}}$ *for every* $\bar{\mathbf{U}}\in\mathrm{UPS}(\mathbf{B})$.

Proof of Lemma 2 This is a slight variation of a well-known result relating Pareto efficiency to linear social welfare functions (see for example Proposition 16.E.2, pg.560, in Mas-Colell et al., 1995). The statement in terms of *strict* system of weights is valid because the shape of $\mathrm{UPS}(\mathbf{B})$ is a simplex, not simply a convex set. ■

Since there is no possibility of confusion we omit the dependence of $\mathbf{E}$ on $\mathbf{B}$. Observe that for any allocation $\mathbf{c}$, the vector of utilities is $\mathbf{u}(\mathbf{c})=\sum_{i=1}^{n}c_i\mathbf{e}^i$. If there exists a strict system of weights $\boldsymbol{\mu}$ and a strictly positive constant κ such that $\boldsymbol{\mu}\cdot\mathbf{e}^i=\kappa$ we have that for any allocation $\mathbf{c}$

$$\boldsymbol{\mu}\cdot\mathbf{u}(\mathbf{c})=\sum_{i=1}^{n}c_i\left(\boldsymbol{\mu}\cdot\mathbf{e}^{(i)}\right)=\kappa\sum_{i=1}^{n}c_i$$

Since $\kappa>0$, we have that $\boldsymbol{\mu}\cdot\mathbf{u}(\mathbf{c})$ is maximal whenever $\sum_{i=1}^{n}c_i=1$. Hence, any allocation such that $\sum_{i=1}^{n}c_i=1$ is Pareto efficient and the economy is regular.

Now, suppose any allocation such that $\sum_{i=1}^{n}c_i=1$ is Pareto efficient. Consider an interior allocation, i.e. such that $c_i>0$ for every $i\in\mathcal{N}$. The unique possible strict system of weights that can separate $\mathbf{u}(\mathbf{c})$ to the utility possibility set in the form of lemma 2 is the strict system of weights orthonormal to the hyperplane that contains the n columns of the matrix of network externalities. Obviously, this system of weights also separates $\mathbf{u}(\mathbf{c})$ to the utility possibility set when $c_i=0$ for some $i\in\mathcal{N}$. Hence, we have the unique candidate for the strict system of weights in the statement of proposition 2. From lemma 2 we know that in particular $\boldsymbol{\mu}\cdot\mathbf{u}(\mathbf{c})\geq\boldsymbol{\mu}\cdot\mathbf{u}(0)=0$. We can ensure that in fact this last inequality is strict since if it were equal to zero we would not be in a generic situation.[33]■

[33] If it were equal to zero this would imply that the columns of the matrix of network externalities are linearly dependent, and hence that the determinant of $\mathbf{E}$ is equal to zero. This would mean that we were considering a non solvable system of bilateral influences.

Proof of Proposition 3

From proposition 2 we know that there exists an strict system of weights $\boldsymbol{\mu}$ and a strictly positive constant such that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \kappa$ for every $i \in \mathcal{N}$. In matrix terms this is equal to

$$\mathbf{E}^T(\mathbf{B}) \cdot \boldsymbol{\mu} = \kappa \mathbf{1}$$

where $\mathbf{E}^T(\mathbf{B})$ is the transpose matrix of $\mathbf{E}(\mathbf{B})$ and $\mathbf{1}$ is the n-dimensional vector with all entries equal to 1. Hence, we have that, since the inverse matrix of $\mathbf{E}^T(\mathbf{B})$ is equal to $(\mathbf{I}-\mathbf{B})^T$,

$$\boldsymbol{\mu} = \kappa\left((\mathbf{I}-\mathbf{B})^T \cdot \mathbf{1}\right)$$

Therefore, $\mu_i = \kappa\left(1 - \sum_{j \neq i} b_{ji}\right)$. Since $\boldsymbol{\mu}$ is an strict system of weights, we have that

$$\kappa \sum_{i=1}^{n}\left(1 - \sum_{j \neq i} b_{ji}\right) = 1$$

and hence

$$\kappa = \frac{1}{\sum_{i=1}^{n}\left(1 - \sum_{j \neq i} b_{ji}\right)}$$

Let $\delta_i = 1 - \sum_{j \neq i} b_{ji}$, and let $\delta = \sum_{i=1}^{n} \delta_i$. Then

$$\mu_i = \frac{\delta_i}{\delta}$$

Since $\boldsymbol{\mu}$ is an strict system of weights, all entries of $\boldsymbol{\mu}$ have to be strictly positive, and this can only happen if either all δ_i's are strictly positive or all δ_i's are strictly negative. However, in the latter case κ would be negative, since $\kappa = 1/(\delta)$. Hence to obtain a regular economy it is necessary that $\delta_i = 1 - \sum_{j \neq i} b_{ji} > 0$ for every $i \in \mathcal{N}$.

The sufficiency result is almost immediate. Consider the weights and κ defined in Lemma 1 in the text. Then by construction these coefficients satisfy the regularity condition in proposition 2. ∎

Proof of Proposition 4

Let $\boldsymbol{\mu}$ and κ be the strict system of weights and constant from Proposition 2 associated to the economy. Let $\mathcal{J} \subseteq \mathcal{N}$ be the set of agents for which $d_j \geq u_j^{min}$. The Nash bargaining problem with network externalities is equal to

$$\max_{\mathbf{u} \in \mathrm{UPS}(\mathbf{B})} \sum_{i=1}^{n} \ln(u_i - d_i)$$

subject to

$$\sum_{i=1}^{n} \mu_i u_i = \kappa \tag{9}$$

$$u_i \geq d_i \quad if\ i \in \mathcal{J} \tag{10}$$

$$u_i \geq u_i^{min} \quad if\ i \notin \mathcal{J} \tag{11}$$

We know that the solution to this problem is unique. We denote this solution $\mathbf{u}^s$. Let $\bar{\psi}$ be the multiplier associated to restriction (9). The Kuhn-Tucker conditions of the problem are

$$\frac{1}{u_i^S - d_i} \leq \bar{\psi}\mu_i \qquad with\ equality\ if\ u_i^S > d_i \quad (i \in \mathcal{J}) \tag{12}$$

$$\frac{1}{u_i^S - d_i} \leq \bar{\psi}\mu_i \qquad with\ equality\ if\ u_i^S > u_i^{min} \quad (i \notin \mathcal{J}) \tag{13}$$

From (12) we obtain that for each $i \in \mathcal{J}$ we must have $\frac{1}{u_i^S - d_i} = \psi\mu_i$. If not, the value of the objective function in the solution would be $-\infty$. Hence, if, for simplicity, we denote $\psi = 1/\bar{\psi}$, we have

$$u_i^S = d_i + \psi\frac{1}{\mu_i} \qquad for\ every\ i \in \mathcal{J}$$

On the other hand, we obtain from (13) that, for every $i \notin \mathcal{J}$, u_i^S must satisfy

$$u_i^S = \max\left\{u_i^{min}, d_i + \psi\frac{1}{\mu_i}\right\} \quad for\ every\ i \notin \mathcal{J}$$

Observe in particular that, for every $i \notin \mathcal{J}$ it holds that $u_i^S = u_i^{min}$ if and only if $\mu_i\left(u_i^{min} - d_i\right) \geq \psi$. Using this fact, we proceed to provide and algorithm that at most in n steps provides the solution to the problem. As we stated in text, we suppose without loss of generality that $\mu_1\left(u_1^{min} - d_1\right) \geq \cdots \geq \mu_n\left(u_n^{min} - d_n\right)$

Step 0:
Suppose $u_i^S = d_i + \psi^{(0)}\frac{1}{\mu_i}$ for every $i \in \mathcal{N}$. The multiplier $\psi^{(0)}$ is equal to $\psi^{(0)} = \frac{1}{n}\left(\sum_{i=1}^n \mu_i\left(u_i^S - d_i\right)\right) = \frac{1}{n}\left(\kappa - \boldsymbol{\mu}\cdot\mathbf{d}\right)$. If $\mu_1\left(u_1^{min} - d_1\right) < \psi^{(0)}$, then $\psi = \psi^{(0)}$ and $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done. If not, go to step 1.

Step 1:
Suppose $u_1^S = u_1^{min}$ and $u_i^S = d_i + \psi^{(1)}\frac{1}{\mu_i}$ for every $i > 1$. The multiplier $\psi^{(1)}$ is equal to $\psi^{(1)} = \frac{1}{n-1}\left(\sum_{i=2}^n \mu_i\left(u_i^S - d_i\right)\right) = \frac{1}{n-1}\left(\kappa - \boldsymbol{\mu}\mathbf{d} - \mu_1\left(u_1^S - d_1\right)\right)$. Observe that

$$(n-1)\psi^{(1)} = \kappa - \boldsymbol{\mu}\mathbf{d} - \mu_1\left(u_1^S - d_1\right) \leq n\psi^{(0)} - \psi^{(0)}$$

Hence, $\psi^{(1)} \leq \psi^{(0)}$, and therefore we know for sure that $\mu_1\left(u_1^{min} - d_1\right) \geq \psi^{(1)}$. If $\mu_2\left(u_2^{min} - d_2\right) < \psi^{(1)}$, then $\psi = \psi^{(1)}$ and $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done. If not, go to step 2.

Step k $(2 \leq k < n)$:

Suppose $u_i^S = u_i^{min}$ for $i \leq k$ and $u_i^S = d_i + \psi^{(k)} \frac{1}{\mu_i}$ for $i > k$. An analogous reasoning to the one in the previous step establishes that $\psi^{(k)} \leq \psi^{(k-1)}$. In fact

$$(n-k)\,\psi^{(k)} = \kappa - \boldsymbol{\mu} \cdot \mathbf{d} - \sum_{l=1}^{k} \mu_l \left(u_l^S - d_l\right) \leq (n-k+1)\,\psi^{(k-1)} - \psi^{(k-1)} = (n-k)\,\psi^{(k-1)}$$

The last inequality follows from the previous step of the procedure. If $\mu_{k+1}\left(u_{k+1}^{min} - d_{k+1}\right) < \psi^{(k)}$, $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done.

This process finishes at most in step $n-1$ since in this case we get that

$$\psi^{(n-1)} = \kappa - \boldsymbol{\mu}\mathbf{d} - \sum_{i=1}^{n-1} \mu_i \left(u_i^{min} - d_i\right) > \mu_n \left(u_n^{min} - d_n\right)$$

This last inequality follows from the fact that $\kappa = \boldsymbol{\mu} \cdot \mathbf{u}^S > \boldsymbol{\mu} \cdot \mathbf{u}^{min}$, since $\mathbf{u}^{min}$ can not be the total vector of utilities associated to an efficient allocation. Thus, if we arrive to step $n-1$, we can ensure that the utility vector associated to the Nash bargaining solution is $u_i^S = u_i^{min}$ for $i < n$ and $u_n^S = d_n + \psi^{(n-1)} \frac{1}{\mu_n} > u_n^{min}$. ∎

Proof of Proposition 5

The matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

The matrix of network externalities of the whole economy is $\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1}$ and we can decompose it as follows:

$$\mathbf{E}(\mathbf{B}) = (\mathbf{I}_n - \mathbf{B})^{-1} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n_1} - \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{I}_{n_2} - \mathbf{B}_{22} \end{pmatrix}^{-1}$$

To avoid misunderstandings, we omit the dependence on $\mathbf{B}$ for $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_{12}$ and $\mathbf{E}_{21}$. In particular, the following two conditions are satisfied:

$$\mathbf{E}_1 \cdot (\mathbf{I}_{n_1} - \mathbf{B}_{11}) + \mathbf{E}_{12} \cdot \mathbf{B}_{21} = \mathbf{I}_{n_1} \tag{14}$$

$$\mathbf{E}_1 \cdot \mathbf{B}_{12} + \mathbf{E}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22}) = \mathbf{0}_{n_1} \tag{15}$$

From the second condition, we obtain that

$$\mathbf{E}_{12} = -\mathbf{E}_1 \cdot \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1}$$

Plugging this back into the first condition we obtain that

$$\mathbf{E}_1 \cdot \left(\mathbf{I}_{n_1} - \mathbf{B}_{11} - \mathbf{B}_{12} \cdot \left(\mathbf{I}_{n_2} - \mathbf{B}_{22} \right)^{-1} \mathbf{B}_{21} \right) = \mathbf{I}_{n_1}$$

And the result follows. ∎

Proof of proposition 6

Fix α. The network in which the minimal utility of an agent is maximal is the complete network, where all pair of agents are connected. The matrix $\mathbf{E}$ for the complete network has entries $\frac{1-(n-2)\alpha}{(1+\alpha)(1-(n-1)\alpha)}$ in the diagonal and $\frac{\alpha}{(1+\alpha)(1-(n-1)\alpha)}$ outside the diagonal. The minimal utility an agent can obtain is the minimum of this two numbers, which coincides with the coefficient outside the diagonal, given the regularity assumption $1-(n-1)\alpha > 0$. Hence for any α-economy $\mathbf{B}$ we have that

$$u_i^{min}(\mathbf{B}) \leq \frac{\alpha}{(1+\alpha)(1-(n-1)\alpha)} \tag{16}$$

If $\mathbf{d} = \mathbf{0}$ we have that the condition to stop in the first step of the algorithm provided in the proof of Proposition 4 is

$$(1 - deg_i(\mathbf{B})\alpha)\, u_i^{min}(\mathbf{B}) \leq \frac{1}{n} \tag{17}$$

Given the regularity condition we know that $\alpha/(1+\alpha) < 1/n$ and therefore

$$\frac{\alpha}{(1+\alpha)(1-(n-1)\alpha)} < \frac{1}{n(1-deg_i(\mathbf{g})\alpha)} \tag{18}$$

Hence the condition in the first step of the algorithm provided in the proof of Proposition 4 is satisfied, and we are done. ∎

Proof of Proposition 7

We can rewrite the total utility an agent obtain in an interior solution as

$$u_i^s = d_i + \frac{1}{n\delta_i}\left(1 - \sum_{j=1}^{n} \delta_j d_j\right) \tag{19}$$

If $i \neq j \neq k$, straightforward calculus yields to

$$\frac{\partial u_i^s}{\partial b_{kj}} = \frac{d_j}{n\delta_i} \tag{20}$$

and the result of the first part of the proposition follows, since in any regular economy $\delta_i > 0$ for all $i \in \mathcal{N}$.

If $i \neq j$ we have that

$$\frac{\partial u_i^s}{\partial b_{ji}} = \frac{1}{n\delta_i^2}\left(1 - \sum_{k \neq i} \delta_k d_k\right) \tag{21}$$

Again, by the regularity condition, the result follows. ∎

Proof of Proposition 8

When $\mathbf{d} = \mathbf{0}$ the share agent i obtains in the Nash bargaining solution when it is interior is

$$c_i^s = \frac{1}{n\delta_i} - \sum_{j \neq i} b_{ij} \frac{1}{n\delta_j} \tag{22}$$

Let $k \neq i$. If we differentiate the expression in (22) with respect to b_{ki} we obtain

$$\frac{\partial c_i^s}{\partial b_{ki}} = \frac{1}{n\delta_i^2} > 0$$

and the first part of the proposition follows.

If i, j and k are pairwise different we have that

$$\frac{\partial c_i^s}{\partial b_{kj}} = -\frac{b_{ij}}{n\delta_j^2}$$

Hence, $\frac{\partial c_i^s}{\partial b_{kj}} b_{ij} \leq 0$ with equality if and only if $b_{ij} = 0$.

Finally, if $i \neq j$ we have that

$$\frac{\partial c_i^s}{\partial b_{ij}} = -\frac{1}{n} \left(\frac{1}{\delta_j} + b_{ij} \frac{1}{\delta_j^2} \right)$$

Hence,

$$\frac{\partial c_i^s}{\partial b_{ij}} > 0 \Leftrightarrow \delta_j < -b_{ij}$$

∎

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